Differential geometry of color space

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1. Introduction

A theoretical model of the colour perception process has been proposed in previous works[1,2]. When the notion of statistical fluctuations in a probability space is added to the previous model, a metric tensor, which measure small differences in the colour representation space, is obtained. The integration of this metric tensor over the geodesic lines provides the generalized statistical distance. It makes it possible to determinate large differences in the colour space. The Riemannian space generated by the generalized statistical distance is called *Ñ space*. A colour stimulus is represented by a unique point in the *Ñ surface*. In this contribution we introduce the Ñ space and study the corresponding differential geometry and the equation of the geodesic lines over the Ñ surface.

Taking into account the statistical nature of the interaction between radiation and the fundamental mechanisms of the visual system, it is possible to assign to any colour stimulus a set of probabilities $\vec{p} = (p_1, p_2, p_3)$, where p_i is the probability of interaction of a spectral power distribution $\rho(\lambda)$ with the i-th fundamental mechanism. These probabilities can be computed by taking into account a given set of colour-matching functions and distribution $\rho(\lambda)$[1]. In this way, a colour stimulus can be represented by a point $\vec{p}$ in the probability space P_3. Any colour stimulus has associated a point over the plane S_P, given by condition

$$\sum_{i=1}^{3} p_i = 1 . \qquad (1)$$

When the statistical fluctuations are considered, is it possible to define in this space the metric tensor

$$ds_g^2 = \frac{1}{4}\left[\sum_{i=1}^{3}\frac{(dp_i)^2}{p_i(1-p_i)}\right]^{1/2}, \qquad (2)$$

where ds_g provides the distance between two neighbour colour stimuli represented by probabilities $\vec{p}$ and $\vec{p}+d\vec{p}$. It becomes obvious that tensor (2) provides a Riemannian metric but not an Euclidean one. In this situation to determinate the distance in the probability space is a difficult task. We now establish a new representation space defined by the change of variable

$$x_i = \sin^{-1}\left(\sqrt{p_i}\right) \qquad \text{(i=1, 2, 3).} \qquad (3)$$

We will refer to this space as *Ñ space*. Any colour stimulus can be represented in this space by a point $\vec{x} = (x_1, x_2, x_3)$.

With the change of variable (3), the metric tensor (2) can be rewritten as

$$ds_g^2 = \sum_{i=1}^{3} (dx_i)^2 , \qquad (4)$$

in such a way that the distance in the Ñ space becomes the Euclidean one. By introducing the change of variable (3) in equation (1) we obtain condition

$$\sum_{i=1}^{3} \sin^2(x_i) = 1 , \qquad (5)$$

which represents a surface in the new representation space. Of course this surface is not a plane as in the probability space, and we will refer to it as *Ñ surface*. All the colour stimulus can be represented by a point belonging to the positive quadrant of this surface. The Ñ surface is shown in Figure 1.

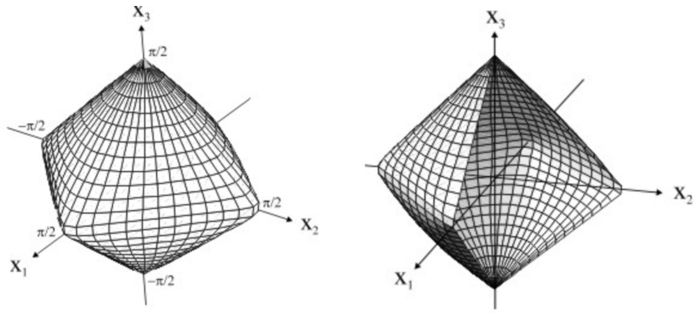

Figure 1: Ñ surface. All the possible colour stimuli are contained in the positive part

The distance between two colour stimulus $\vec{x}_1$ and $\vec{x}_2$ is given by the length of the geodesic line joining both points. It can be demonstrated that over the Ñ surface there is a unique geodesic line joining two points[3]. The three Euler-Lagrange equations which the geodesic lines must satisfy are given by

$$2x_i'' + \sin(2x_i) = 0\,. \tag{6}$$

The solution of this system is that of three coupled pendula and it provides the following value of the distance between two points $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$:

$$s_g = \frac{1}{\sqrt{2E_i}}\left|F_i\left(x_i^{(2)},\frac{1}{\sqrt{2E_i}}\right) - F_i\left(x_i^{(1)},\frac{1}{\sqrt{2E_i}}\right)\right|, \tag{7}$$

where

$$\sum_{i=1}^{3} E_i = 1, \tag{8}$$

$$F_i(a,q) = \int_0^a \frac{dx_i}{\sqrt{1-q^2\sin^2(x_i)}} \tag{9}$$

are the first kind elliptic integrals, and E_i is the "energy" of the i-th pendulum. Quantities E_i should be determined in the integration process in order to satisfy equations (7).

References

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3. W.M. Boothby, "An introduction to differentiable manifolds and riemannian geometry" (Academic Press, London 1986).