Least Squares Method for Computing Weighting Tables for Calculating CIE Tristimulus Values

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ABSTRACT

Weighting tables are used for calculating CIE tristimulus values. In this paper, firstly, it was found that the optimum weights satisfy an overdetermined linear system of equations. Secondly, the least squares method is considered for solving the linear system and the least squares solution can be obtained by solving a tri-diagonal, symmetric and positive definite linear system of equations. Finally, comparisons between the weighting tables computed by the proposed method and other weighting tables such as ASTM Intl. Table 5, and optimum weighting tables by Li et al's are given. The results clearly show that the newly developed weighting tables perform equally well or better than the other types of weighting tables for 10nm interval. While for the 20nm interval, the optimum weighting table given by Li et al is still the best

1. INTRODUCTION

It is often required by industry that weights (weighting table) $W_{V,i}^{(\Delta\lambda)}$ are prepared in advance and tristimulus values can be directly computed using the measured reflectance values R_i' at $\Delta\lambda$ nm interval (see equation (1)).

$$V' = \sum_{i=0}^{n} W_{V,i}^{(\Delta\lambda)} R_i' \ , \text{ with } V = X \ , \ Y \text{ and } Z \tag{1}$$

where $W_{V,i}^{(\Delta\lambda)}$ depends on the spectral power distribution (SPD) and the CIE colour matching functions (CMF) used. The CIE has never provided guidelines to calculate weighting tables, and various discrepant methods have been used. Hence it is possible to obtain significantly different tristimulus values from the same set of spectral data. The differences are normally quite small, but can sometimes be significant (over 1 CIELAB ΔE units) and introduce errors that are completely avoidable. For overcoming this problem, the ASTM Intl. [1] has published two sets of weighting tables known as Table 5 and Table 6 respectively. However in practice, the illuminant required may be different from the standard and users have to prepare their own weighting tables corresponding to the illuminant actually used. ASTM Intl. E2022-99 [2] provided a standard calculation method to generate weighting tables of Table 5 for a non-standard illuminant. Recently, a direct method [3] is given for generating the optimum weighting tables. Numerical comparisons showed that the optimum weighting table is more accurate than both ASTM Intl. weighting tables. This paper describes a new method for computing the optimum weights referred as the **least squares weights** (**weighting table**). Its performance is compared with other existed weighting tables.

2. THE NEW METHOD

Ideally, the weights $W_{V,i}^{(\Delta\lambda)}$ should be chosen so that the difference between V' and V defined by eq. (2) below is as small as possible.

$$V = k\int_{a}^{b} S(\lambda)\bar{v}(\lambda)R(\lambda)d\lambda \text{ with } v = x \ , \ y \text{ and } z \ . \tag{2}$$

Here $S(\lambda)$ is the SPD of an illuminant, $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ are the CIE standard colour matching functions (CMF), and $R(\lambda)$ is the reflectance function of the colour. (a,b) is the visible range of

wavelengths and recently CIE [4] recommended that a = 380nm and b = 780nm, and k is the normalizing factor. Note that $R_i^{'}$ satisfy [3,5]

$$R_i^{'} = \int_a^b P_i(\lambda) R(\lambda) d\lambda \tag{3}$$

where $P_i(\lambda)$ is the instrumental (spectral bandpass) function and has a triangular shape with a half-height equal to the wavelength interval. Besides [3,5-6], it is scaled so that

$$\int_a^b P_i(\lambda) d\lambda = 1 \tag{4}$$

Therefore, $P_i(\lambda)$ has the expression

$$P_i(\lambda) = \begin{cases} (\lambda - \lambda_{i-1})/(\Delta\lambda)^2, & \text{when } \lambda_{i-1} \le \lambda \le \lambda_i \\ (\lambda_{i+1} - \lambda)/(\Delta\lambda)^2, & \text{when } \lambda_i \le \lambda \le \lambda_{i+1} \\ 0, & \text{when } \lambda < \lambda_{i-1} \text{ or } \lambda > \lambda_{i+1} \end{cases} \tag{5}$$

Thus the difference between $V^{'}$ and V is given by the following:

$$V^{'} - V = \int_a^b D(\lambda) R(\lambda) d\lambda \tag{6}$$

where $D(\lambda)$ is given by

$$D(\lambda) = [\sum_{i=0}^{n} P_i(\lambda) W_{V,i}^{(\Delta\lambda)}] - W_V(\lambda) \ , \text{ with } \quad W_V(\lambda) = kS(\lambda)\bar{v}(\lambda) \tag{7}$$

Since

$$V^{'} - V = \int_a^b D(\lambda) R(\lambda) d\lambda \approx \sum_{\lambda=a,\ \text{step } 1nm}^{b} D(\lambda) R(\lambda)$$

therefore, the difference is zero or reasonable small if

$$D(\lambda) = 0 \,, \quad \text{for } \lambda = a, a+1, a+2, \cdots, b$$

or equivalently

$$\sum_{i=0}^{n} P_i(\lambda) W_{V,i}^{(\Delta\lambda)} = W_V(\lambda) \,, \quad \text{for } \lambda = a, a+1, a+2, \cdots, b \tag{8}$$

Note that eq. (8) is a linear system of equations. It has 401 equations because $a = 380$ nm and $b = 780$ nm. Since $\Delta\lambda$ is greater than 1mm, the number of the unknowns $W_{V,i}^{(\Delta\lambda)}$ of the system is much less than 401. For example, if $\Delta\lambda = 10$ nm, it has 41 unknowns with $n = 40$ and if $\Delta\lambda = 20$ nm, it has 21 unknowns with $n = 20$. Thus, the linear system (8) is an overdetermined system of equations [7]. It may have no solution in the normal sense. However, it [7] always has a least squares solution. If we let

$$u_V^{(\Delta\lambda)} = \begin{pmatrix} W_{V,0}^{(\Delta\lambda)} \\ W_{V,1}^{(\Delta\lambda)} \\ \vdots \\ W_{V,n}^{(\Delta\lambda)} \end{pmatrix}, \quad g_V = \begin{pmatrix} W_V(a) \\ W_V(a+1) \\ \vdots \\ W_V(b) \end{pmatrix}, \quad P = \begin{pmatrix} q & s & & & \\ & q & \ddots & \ddots & \\ & & \ddots & \ddots & s \\ & & & q & s \\ & & & & t \end{pmatrix} \tag{9}$$

$$q = \frac{1}{(\Delta\lambda)^2} \begin{pmatrix} \Delta\lambda \\ \Delta\lambda - 1 \\ \vdots \\ 1 \end{pmatrix}, \quad s = \frac{1}{(\Delta\lambda)^2} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \Delta\lambda - 1 \end{pmatrix} \quad t = \frac{\Delta\lambda}{(\Delta\lambda)^2} = \frac{1}{\Delta\lambda} \tag{10}$$

then the lest squares solution of the overdetermined linear systems of equations (8) can be obtained by solving the following tri-diagonal normal equation:

$$B u_V^{(\Delta\lambda)} = h_V \tag{11}$$

where

$$B = P^T P = \begin{pmatrix} q^T q & q^T s & & & \\ q^T s & q^T q + s^T s & \ddots & & \\ & \ddots & \ddots & q^T s & \\ & & q^T s & q^T q + s^T s & q^T s \\ & & & q^T s & s^T s + t^2 \end{pmatrix}, \quad h_V = P^T g_V \tag{12}$$

Here the superscript T means the transpose of a vector or matrix. By the definitions of vectors q, s, and the quantity t, it can be shown that

$$q^T s = \frac{(\Delta\lambda)^2 - 1}{6(\Delta\lambda)^3},\ q^T q = \frac{(\Delta\lambda + 1)(2\Delta\lambda + 1)}{6(\Delta\lambda)^3},\ s^T s = \frac{(\Delta\lambda - 1)(2\Delta\lambda - 1)}{6(\Delta\lambda)^3} \tag{13}$$

$$q^T q + s^T s = \frac{2(\Delta\lambda)^2 + 1}{3(\Delta\lambda)^3},\quad s^T s + t^2 = q^T q \tag{14}$$

Since the $401\times(n+1)$ matrix P defined by eq. (9) has a full column rank, therefore, the matrix B is symmetric and positive definite. Thus, the normal equation (11) can be easily solved. In addition, the coefficient matrix B does not change with the change of V. Note that g_X (g_Y, g_Z) is the 1nm weights for computing X (Y, Z). Thus, if we let $W^{(\Delta\lambda)} = [u_X^{(\Delta\lambda)}, u_Y^{(\Delta\lambda)}, u_Z^{(\Delta\lambda)}]$ denote the matrix of the wanted $\Delta\lambda$ nm weighting table and $W^{(1)} = [g_X, g_Y, g_Z]$ be the matrix of the 1nm weighting table, then we have:

$$W^{(\Delta\lambda)} = B^{-1} P^T W^{(1)} \tag{15}$$

Note that the above method for computing the least squares weighting tables is simple and has the same complexity as the one given by Li et al [3].

3. COMPARISON OF DIFFERENT WEIGHTING TABLES

The reflectance functions [8] of 1nm interval data measured from 1269 matt Munsell color chips are used as a standard reflectance set. The 10-nm and 20-nm reflectance function of each samples was calculated from the standard 1nm reflectance set by assuming an ideal instrument [3,5]. The 1nm SPDs of D65, A, F_{BB} (broad band fluorescent lamp) and F_{TB} (three band) together with two sets of CMFs are used. Three types of 10-nm and 20-nm weighting tables were compared under the above four illuminants and the 1931/1964 CMFs. Type 1 tables were ASTM Intl.Table 5 computed using the procedure of ASTM Intl. E2022 [2]. Type 2 tables were those computed by using the method of Li et al [3]. Type 3 tables were those derived by this study. They are designated as **T5**, **OP**, and **LS Tables** respectively.

Note that the ASTM weighting tables of Table 6 is not considered here, since it involves the Venable weighting table and there is no standard procedure for generating the Venable weighting tables. Besides, the T5 table must be used for the reflectance corrected using the Stearns and Stearns formula. While the OP and LS tables must be used for reflectance without bandpass correction.

Three sets of tristimulus values were obtained by using the 1-nm, 10-nm, and 20-nm weighting tables and reflectances. The 1nm tristimulus values were taken as standard and were used to compute the **CIELAB** colour differences ΔE_{ab}^* against the 10-nm and 20-nm tristimulus values. The median and maximum ΔE_{ab}^* were used for measuring the performances of each weighting table. A smaller colour differences indicate a better performance of the weighting table considered. Since the colour differences are not normally distributed, the median will generally reflect the average performance of each table. Besides, we also interest in the maximum colour difference, which is the worse case when the corresponding weighting table is used. The results at 10 and 20 nm intervals with the two standard colorimetric observers being combined together are listed in Tables 1 and 2 respectively. The values in bold are the best cases (smallest median or maximum ΔE_{ab}^*) in each column. From each table and using any of the two measures, it can be seen the T5 table performed the worst. While for the LS and OP weighting tables, the LS table performed better for the continuous illuminants for the 10nm table, the opposite is true for the fluorescent illuminants. Generally, they are competitive for the 10nm weighting table. As for the 20nm weighting table, the OP table performed better than the LS table in general. Besides, from the 10nm weighting tables, the OP and LS tables performed slightly better than the T5 tables. Using any of the three tables, the error introduced is less than 0.01 ΔE_{ab}^* units for the continuous illuminants and 0.1 units for the fluorescent illuminants. The maximum error is less than 0.03 ΔE_{ab}^* units for the continuous illuminants and 0.5 units for the fluorescent illuminants. While from 20 nm weighting tables, the OP and LS tables are much better

than the T5 table. Using T5 tables may introduce errors of 0.15 and 0.5 ΔE^*_{ab} units for the continuous and fluorescent illuminants respectively. The error may be as large as 1.8 ΔE^*_{ab} units under the fluorescent illuminants. Using OP tables may introduce errors of 0.02 and 0.35 ΔE^*_{ab} units for the continuous and fluorescent illuminants respectively. The error may be as large as 0.9 units under the three band fluorescent illuminants.

4. CONCLUSIONS

A method for computing the Least Squares weighting tables was derived. The method is as simple as the one given by Li et al [3]. The results from numerical simulations demonstrated that the Least Squares weights computed by the newly developed algorithm performed equally well as or better than the Optimum Table [3] at 10 nm interval. While for 20-nm weighting tables, the Optimum Weighting Table is better than the Least Squares Table. In both 10 nm and 20 nm cases, the ASTM Intl. Table 5 performed the worst.

	Median				Maximum			
	D65	A	F_{BB}	F_{TB}	D65	A	F_{BB}	F_{TB}
Table 5	0.0060	0.0049	0.0164	0.0600	0.0203	0.0164	0.1592	0.5241
Optimum	0.0019	0.0016	**0.0151**	**0.0512**	0.0084	0.0064	**0.1449**	**0.4906**
Least Squares	**0.0017**	**0.0008**	0.0159	0.0533	**0.0075**	**0.0035**	0.1460	0.4915

Table 1: Performance of each table at 10nm interval under the combination of the two observers in terms of Median and Maximum CIELAB colour difference

	Median				Maximum			
	D65	A	F_{BB}	F_{TB}	D65	A	F_{BB}	F_{TB}
Table 5	0.1525	0.1433	0.2468	0.5287	0.3256	0.2602	0.7047	1.7632
Optimum	**0.0185**	**0.0177**	**0.0825**	**0.3329**	**0.0535**	**0.0414**	0.3931	**0.8844**
Least Squares	0.0344	0.0435	0.0990	0.3676	0.0559	0.0531	**0.3855**	0.9755

Table 2: Performance of each table at 20nm interval under the combination of the two observers in terms of Median and Maximum CIELAB colour difference

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