Theorem and Formula of Subtractive Color Mixture

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ABSTRACT
This paper provides some theorems related to subtractive color mixture, and using these theorems, finally it is shown that the centroid of resultant tristimulus values is represented approximately in the multiplication formula between component tristimulus values.

1. INTRODUCTION

In additive color mixture, when more than two component tristimulus values are given then the mixture result is determined from a linear combination of the components. On the other hand, in subtractive color mixture, the resultant tristimulus values are not uniquely determined from component tristimulus values which are not clarified also in today.

This paper provides some theorems related to subtractive color mixture, and using these theorems, finally it is shown that the centroid of resultant tristimulus values is represented approximately in the multiplication formula between component tristimulus values. This formula corresponds to the addition in additive color mixture and has a significant meaning in the research history of color mixture.

2. THEOREM AND FORMULA

Notation and definition

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\bar{x}} = \int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda) \bar{x}(\lambda) d\lambda \ ,$$

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\bar{x},\max} = \max_{\rho_i(i=1,2,\cdots,n)} \left[\int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda) \bar{x}(\lambda) d\lambda \right],$$

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\bar{x},\min} = \min_{\rho_i(i=1,2,\cdots,n)} \left[\int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda) \bar{x}(\lambda) d\lambda \right] .$$

where

n : number of colorants.

Define a stimulus value for each colorant layer as follows:

$$X_i = \langle \rho_i(\lambda) \rangle_{\bar{x}} \ (i = 1,2,\cdots,n). \tag{1}$$

Equation (1) is the defining constraint of the whole problem, and $X_i \ (i=1,2,\cdots,n)$ are inputs that define the constraints. These X_i values can be chosen only in the interval $X_i \in [0, X_0]$, where X_0 is just the illuminant X value.

<u>Relations in the spectral transmittance of *n* colorants[1]</u>

- Included relation in the transmittance of *n* colorants

The included relations of transparent bands are defined as the cases that a wavelength region of $\rho = 1$ of a colorant is included in wavelength regions of $\rho = 1$ of other colorants larger than the region.

- Separated relation in the transmittance of *n* colorants

The separated relations of absorption bands are defined as the cases that a wavelength region of $\rho = 0$ of a colorant is separated with wavelength regions of $\rho = 0$ of other colorants.

The following theorems and formulas are provided.

[Theorem 1][1]: Theoretical maximum bound

$$\langle \rho_1(\lambda) \rho_2(\lambda) \rangle_{\bar{x},\max} = \min[X_1, X_2] . \tag{2}$$

[Theorem 2][1,2] : Theoretical minimum bound

If $X_0 < X_1 + X_2$, then $\langle \rho_1(\lambda)\rho_2(\lambda) \rangle_{\bar{x},\min} = X_1 + X_2 - X_0$, else if $X_1 + X_2 \le X_0$, then $\langle \rho_1(\lambda)\rho_2(\lambda) \rangle_{\bar{x},\min} = 0$. (3)

[Theorem 3][2] : Relation between the maximum bound and the minimum bound

$$\langle(1-\rho_1(\lambda))\rho_2(\lambda)\rangle_{\bar{x},\max} = X_2 - \langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\min} \ , \tag{4}$$

[Theorem 4][2] : Relation between the minimum bound and the maximum bound

$$\langle(1-\rho_1(\lambda))\rho_2(\lambda)\rangle_{\bar{x},\min} = X_2 - \langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\max} \ . \tag{5}$$

[Theorem 5]

Assume $X_2 < X_1$. $\rho_1(\lambda)$ of $X_1 = \langle\rho_1(\lambda)\rangle_{\bar{x}}$ and $1-\rho_1(\lambda)$ are considered. In this case, the centroid of $\langle\rho(\lambda)\rho_2(\lambda)\rangle_{\bar{x}}$ is $X_2/2$, where $\rho(\lambda)=\rho_1(\lambda)$ or $\rho(\lambda)=1-\rho_1(\lambda)$. In the same way for the exchanged relation of $X_1 \le X_2$, the theorem is valid in the exchanged form in which the centroid is $X_1/2$.

Proof

Consider the case of $X_1 + X_2 < X_0$ (non-existence of *separated relation*).

For $\rho_1(\lambda)$, the following relations are consistent from Theorems 1 and 2.

$$\langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\min} = 0 \ , \tag{6.a}$$

$$\langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\max} = X_2 \ . \tag{6.b}$$

For $1-\rho_1(\lambda)$, the following relations are consistent from Theorems 1 through 4.

$$\langle(1-\rho_1(\lambda))\rho_2(\lambda)\rangle_{\bar{x},\max} = X_2 - \langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\min} = X_2 \ , \tag{7.a}$$

$$\langle(1-\rho_1(\lambda))\rho_2(\lambda)\rangle_{\bar{x},\min} = X_2 - \langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\max} = 0 \ . \tag{7.b}$$

$\rho_1(\lambda)$ and $1-\rho_1(\lambda)$ are in the symmetrical relation, and considering the second equations of Eqs.(7.a) and (7.b) which are inverted along $[0, X_2]$, it can be said that the probability density $\Pr.(X)$ of $\{\rho_1(\lambda)\}$ satisfying $\langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x}} = X$ $(X \in [0, X_2])$ in the min-max range between Eqs.(6.a) and (6.b) is in the symmetrical relation with the probability density $\Pr.'(X')$ of $\{1-\rho_1(\lambda)\}$ satisfying $\langle(1-\rho_1(\lambda))\rho_2(\lambda)\rangle_{\bar{x}} = X'$ $(X' \in [0, X_2])$ in the min-max range of Eqs.(7.a) and (7.b), in other words, $\Pr.(\Delta X) = \Pr.'(X_2 - \Delta X) = \Pr.'(X')$ $(0 \le \Delta X \le X_2)$. Where { } indicates a set. Hence, the centroid averaged in the min-max ranges of $[0, X_2]$ using the probability densities of $\Pr.(X)$ and $\Pr.'(X')$ corresponds to $X_2/2$. Figure 3 shows an example for understanding the relation.

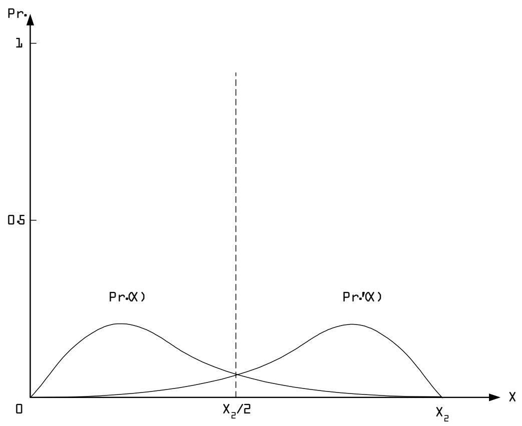

Figure 3

Consider the case of $X_0 \le X_1 + X_2$ (existence of *separated relation*).

For $\rho_1(\lambda)$, the following relation is consistent from Theorems 1 and 2.

$$\langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\min} = X_1 + X_2 - X_0 \ , \tag{8.a}$$

$$\langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\max} = X_2 \ . \tag{8.b}$$

The length of the min-max range of Eqs.(8.a) and (8.b) is $X_0 - X_1$.

For $1-\rho_1(\lambda)$, the following relation is consistent from Theorems 1 through 4.

$$\langle(1-\rho_1(\lambda))\rho_2(\lambda)\rangle_{\bar{x},\max} = X_2 - \langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\min} = X_0 - X_1 \ , \tag{9.a}$$

$$\langle(1-\rho_1(\lambda))\rho_2(\lambda)\rangle_{\bar{x},\min} = X_2 - \langle\rho_1(\lambda)\rho_2(\lambda)\rangle_{\bar{x},\max} = 0 \ . \tag{9.b}$$

The length of the min-max range between Eqs.(9.a) and (9.b) is also $X_0 - X_1$.

The probability density $\Pr.(X)$ $(X \in [0, X_0 - X_1])$ of $\{\rho_1(\lambda)\}$ in the min-max range between Eqs.(9.a) and (9.b) is in the symmetrical relation with the probability density $\mathrm{Pr}.'(X')$ $(X' \in [X_1 + X_2 - X_0, X_2])$ of $\{1 - \rho_1(\lambda)\}$ in the min-max range of Eqs.(8.a) and (8.b), in other words, $\mathrm{Pr}.(\Delta X) = \mathrm{Pr}.'(X_2 - \Delta X) = \mathrm{Pr}.'(X')$ $(0 \le \Delta X \le X_0 - X_1)$. Hence, the centroid averaged using the probability densities corresponds to $X_2/2$ not depending on the numerical relation between $X_1 + X_2 - X_0$ and $X_0 - X_1$. Figures 4(a)(b) show examples for understanding the relation.

For both cases of $X_1 + X_2 < X_0$ and $X_0 \le X_1 + X_2$, the theorem is proven.

For $X_1 \le X_2$, the centroid is evaluated as $X_1/2$ in the same way.

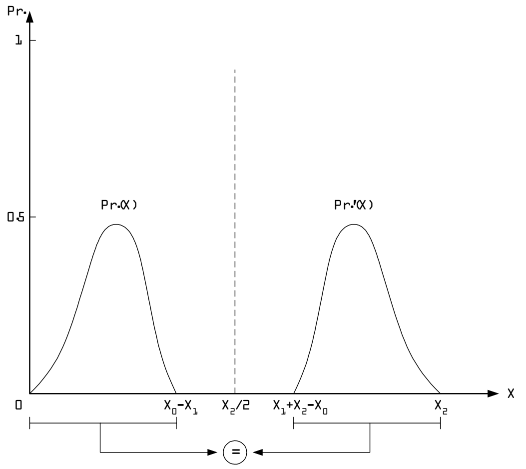

Figure 4(a) $X_0 - X_1 < X_1 + X_2 - X_0$

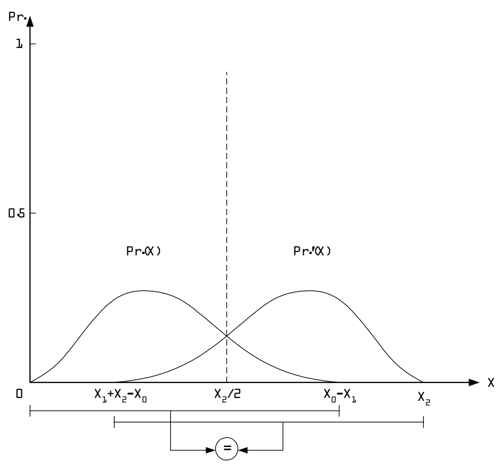

Figure 4(b) $X_1 + X_2 - X_0 \le X_0 - X_1$

[Theorem 6]

Assume $X_2 < X_1$. If $X_1 = X_0/2$, the centroid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ is in the form of $X_1 X_2 / X_0$ in which X_1 and X_2 are multiplied form and X_0 is the normalization factor. In the same way for the exchanged relation of $X_1 \le X_2$, the centroid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ is in the form of $X_1 X_2 / X_0$.

Proof

Consider the case of $X_2 < X_1$. In the framework of Theorem 5, both $\rho_1(\lambda)$ and $1 - \rho_1(\lambda)$ which are in a symmetrical relation related to probability densities are considered. If $X_1 = X_0/2$ then $\{\rho_1(\lambda)\} = \{1 - \rho_1(\lambda)\}$, and only $\rho_1(\lambda)$ should be considered. In this case, the centroid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ is $X_2/2$ for $\rho_1(\lambda)$ from Theorem 5.

Under the assumptions, the following relation is derived, and the theorem is proven.

$$\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x},centroid} = \frac{X_2}{2} = \frac{X_2}{2} \cdot \frac{X_0}{X_0} = \frac{X_0}{2} \cdot \frac{X_2}{X_0} = \frac{X_1 X_2}{X_0} \quad (10)$$

In the same way, the theorem can be proven for $X_1 \le X_2$.

[Theorem 7]

For $X_1 = 0.0$, the centroid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ is 0.0 , and for $X_1 = X_0$, the centroid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ is X_2 , and for both cases, the centroid is described in the form of $X_1 X_2 / X_0$. In the same way for $X_2 = 0.0$ or $X_2 = X_0$.

Proof

In the case of $X_1 = 0.0$, $\rho_1(\lambda)$ is always 0.0 and the centroid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ becomes 0.0 . In the case of $X_1 = X_0$, $\rho_1(\lambda)$ is always 1.0 and the controid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ becomes X_2 . The both centroids satisfy the equation of $X_1 X_2 / X_0$.

[Theorem 8]

$X_1 X_2 / X_0$ is an approximation of the centroid of $\langle \rho_1(\lambda) \cdot \rho_2(\lambda) \rangle_{\bar{x}}$ for given X_1 and X_2 .

Proof

From Theorems 6 and 7, $X_1 X_2 / X_0$ is consistent to three points of $X_1 = 0.0$ (minimum), $X_1 = X_0/2$ (middle) and $X_1 = X_0$ (maximum) (or $X_2 = 0.0$, $X_2 = X_0/2$ and $X_2 = X_0$). Hence, $X_1 X_2 / X_0$ is an approximation for any values of X_1 , X_2 .

For $n(>2)$ colorants, mathematical induction derives the centroid formula of $X_1 X_2 \cdots X_n / X_0^{n-1}$

whose proof is omitted in this article.

3.SIMULATION RESULTS

The color matching functions of CIE1931(1nm) were employed, and numerical calculations were performed in the wavelength range of $[400{,}700]$ (nm). Two hundred of $\rho_i(\lambda)$ randomly selected satisfying a given tristimulus conditions were employed for each $\rho_i(\lambda)$ $(i=1,2,3)$. Equal energy illuminant is assumed in this numerical example. $X_0 = \langle \rho(\lambda)=1 \rangle_{\bar{x}} = 99.81$.

Table 1 shows the results verifying Theorem 8. The errors against $X_1 X_2 / X_0$ are 0.05% for the minimum, 5.67% for the maximum, and 1.77% on average which confirm Theorem 8. The errors were normalized by X_0 . $X_1 X_2 / X_0$ can be calculated easily using indices of the table and can be compared with the values in the table. In the errors, the effect of finite sampling and approximation errors are included.

Table 1

		X_2								
		10	20	30	40	50	60	70	80	90
X_1	10	2.19								
	20	3.58	6.41							
	30	3.71	7.35	10.46						
	40	3.93	8.07	12.42	16.45					
	50	4.39	9.13	14.81	20.42	26.51				
	60	4.70	10.03	17.04	24.47	31.87	39.01			
	70	4.99	11.11	19.79	28.65	37.28	45.84	54.23		
	80	5.84	13.17	23.17	32.84	42.23	51.74	61.29	69.78	
	90	7.89	16.72	26.83	36.73	46.51	56.24	66.14	75.48	83.52

4.CONCLUSIONS

On the basis of the theorems provided in this paper, it was shown that the centroid of resultant tristimulus values is represented approximately in the multiplication form between component tristimulus values. Also simulation results verified the formula. This multiplication corresponds to the addition in additive color mixture and has a significant meaning in the research history of color mixture.

In this paper, only one-dimensional discussions were performed. Tristimulus discussions will be found in references 4 to 6.

References

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